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System for Generating Sequences of Phased Gust or Taxi Loadings

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A method is presented for Monte Carlo generation of phased load conditions for multiple loading durability testing. These load conditions are consistent with the gust and taxi environmental-dynamic model used in aircraft design. Time histories of individual loads from sequences of load conditions are consistent with the load sequences used in standard single load point durability and damage tolerance testing and analysis. Truncation can be applied to a sequence of load conditions so that only the most severe are retained. The truncation process is consistent with the mission analysis approach to design and to durability and damage tolerance. The load conditions can be modified using an induced autocorrelation approach to give variation in the number of zero crossings of the load time histories.

Introduction

LOAD sequences generated for durability and damage tolerance analysis are usually applied at a single point. However, the stresses in a component of structure often arise from combinations of multiple loads. Selection of a single application point and direction is often a difficult task. This is particularly true for aircraft structures flying in turbulence. In that case, loads arise from complex combinations of external and internal forces.

At McDonnell Douglas the full-scale durability test of the C-17 aircraft requires the simultaneous application of test loads which are properly phased by some rational definition of phasing. The following criteria was developed for satisfying this requirement:

1) Time histories for each individual load (e.g., wing root normal bending) must be consistent with load sequences used in durability and damage tolerance testing and analysis.

2) The phasing of the simultaneously applied loads must be consistent with the dynamic gust and taxi models used to design the aircraft.

A method is presented here that meets this criteria. The method generates load conditions, that is, multiple loads for simultaneous application at several points on an aircraft structure. The multiple loads are correlated in conformance with the mathematical model of the gust environment and the structure. In this model the magnitude, frequency response, and correlation of loads are based on a flexible linear aircraft response to a forcing function having the characteristics of a stationary Gaussian random process (see Fig. 1). The specifics of the mathematical model of an aircraft flying in turbulence have been summarized by Hoblit in Ref. 1.

This method as implemented at Douglas Aircraft Corporation is called the Phased Loading Sequence Generator System (PLSGS). This system will be used to generate gust and taxi loadings for the C-17 full-scale durability test. For the C-17 application the taxi forcing function (the runway) was treated as a stationary Gaussian random process, and the aircraft was treated as a flexible linear system. This was necessary for consistency with design durability and damage tolerance analysis, which also made these assumptions. The use of PLSG with these assumptions for taxi may result in the generation of a few load conditions which violate physical constraints of the actual nonlinear system. More exact application of PLSG to taxi loadings would require the reformulation of the probability equations governing the method.

Given the computations of Table 3, everything required for computational implementation of the PLSGS is presented in Table 1 and Eqs. (2), (3), (4), (14), and (18).

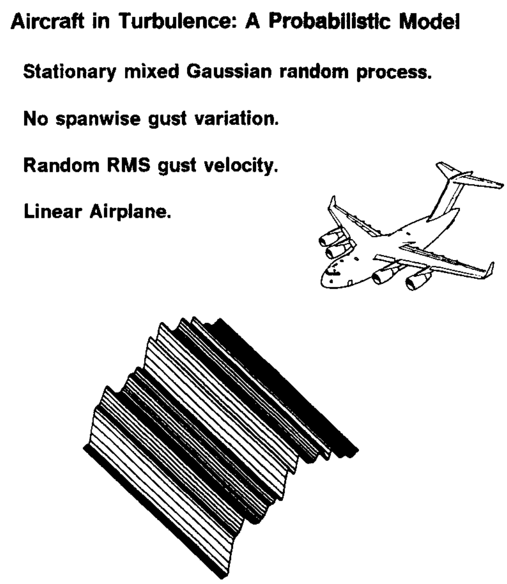

Fig. 1 The gust analysis model.

Table 1 Computational mathematics of PLSGS

Given input
Covariance matrix = Σ_y
Exceedance distribution = $N_0[P_1 e^{-y/\bar{A}b_1} + P_2 e^{-y/\bar{A}b_2}]$
Compute
Random vector = X
Random scaling variable = U
Transformation matrix, T: $TT^* = \Sigma_y$
Load condition vector = $V = UTX$

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Though developed independently, the analysis described here can be considered an extension and application of the work of Dr. R. Noback in Ref. 2.

Phased Loading Sequence Problem for PSD Gust Loads

The problem statement for the phased loading sequence generation system (PLSGS) can be stated as follows: "Generate phased loadings $V = \{v_i\}$ such that the marginal probability distribution for each v_i is consistent with the gust (or taxi) exceedance function and such that specified correlations ρ_{ij} and standard deviation ratios σ_j/σ_i are achieved for all combinations of i, j."

A "phased loading" or "phased load condition" is defined as a multiple of simultaneously acting random loads which possess proper correlation properties. These properties are specified by the parameters ρ_{ij} and σ_j, σ_i whic are defined in Eq. (1). Generation here is defined as Monte Carlo generation, the production of random variables by transformation of pseudo random variables (see Appendix B). General terms from durability and damage tolerance analysis, such as "standard load sequence," and terms from statistical theory are defined in Appendix B.

The atmospheric gust probabilistic model suggests generating a set of random variables $Y_1, Y_2, Y_3, \ldots, Y_N$, and multiplying them by a random scale factor, U, to get the final result. A natural choice of joint distribution for the Y_i is the normal distribution, since that is also the distribution of atmospheric gusts. Taxi loads are also assumed to have a normal distribution in the C-17 application.

The PLSGS consists of the following two step process:

1) Generate Gaussian (normally distributed) phased loadings with covariance matrix equal to that computed in the PSD gust analysis.

2) Multiply each phased loading by a scaling random variable to make each load in the phased loading satisfy the gust exceedance curve.

The technique provides external load sequences over the airplane which are equivalent to the standard load sequences used in durability tests and in damage tolerance analysis. In addition, the loadings comply with the defined requirement of phasing.

Imposition of Phasing

The generation of phased loadings requires a formal definition of phasing which can be used to enable application of accepted statistical theory. The generated loadings are defined to be equivalent to the aircraft environmental loadings if estimates of phasing parameters for the generated loadings converge in probability (with increasing sample size) to the parameters obtained from the aircraft environmental process. That is, generated loading samples are indistinguishable from aircraft environmental loading samples. The generation thus imposes the same mutual dependence properties on the generated loadings as exist in the actual aircraft environment.

The following definition is not only a rational rigorization of the generally held intuitive concept of phasing, it also provides an immediate access to statistical theory. The measure of phasing between two random variables X, Y is defined to be the two parameters

$$\rho_{xy} = \sigma_{xy}/\sigma_x\sigma_y, \qquad \delta_{xy} = \sigma_y/\sigma_x \tag{1}$$

where ρ_{xy} is the correlation between x and y and σ_x, σ_y are the standard deviations of x and y. The traditional confusing convention $\sigma_{xx} = \sigma_x^2$ is here perpetuated. Two loadings are defined to have the same phasing if these parameters are the same for every combination of two loads x, y in each loading. The parameters of Eq. (1) are defined in Appendix B.

The gust model specifies that aircraft gust loads are caused by a mixed Gaussian (normal) random process. A Gaussian random process is completely specified parametrically by its covariance matrix and mean values. In this case the means are all zeroes since gust loads are modeled as incremental loads about a 1-g flight condition. (One-g loads are added for application in test or analysis.) Every Gaussian process of dependent variables is a linear transformation of a process of independent random variables. It is possible to use the covariance matrix Σ_y of the loads to compute a transformation matrix $\boldsymbol{T}$ and transform a vector $\boldsymbol{X}$ of independent random process elements into mutually dependent loads in a loading $\boldsymbol{Y}$. This is true because the symmetric positive definite matrix Σ_y can be written as

$$\Sigma_y = \boldsymbol{E}\Sigma_x\boldsymbol{E}^* \tag{2}$$

where Σ_x is a diagonal matrix with the eigenvalues of Σ_y on the diagonal and $\boldsymbol{E}$ is the matrix of eigenvectors of Σ_y. The matrix

$$\boldsymbol{T} = \boldsymbol{E}\sqrt{\Sigma_x} \tag{3}$$

defines the transformation

$$\boldsymbol{Y} = \boldsymbol{TX} \tag{4}$$

between the mutually dependent elements of $\boldsymbol{Y}$ and the mutually independent unit variance random process elements of $\boldsymbol{X}$ (see Ref. 4). If the elements of $\boldsymbol{X}$ are normally distributed, the elements of $\boldsymbol{Y}$ will also be normally distributed; linear combinations of normal random variables are also normal

Table 2 Imposition of phasing for two unit variance loads

Given Y covariance matrix: $\Sigma_y = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

Solve eigenvalue equation:

$$\det\begin{pmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - \rho^2 = 0$$

For eigenvalues: $\lambda = 1 + \rho, \quad 1 - \rho$

Compute eigenvector matrix: $\boldsymbol{E} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Form X covariance matrix: $\Sigma_x = \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix}$

Compute $\boldsymbol{T}$ transformation:

$$\boldsymbol{T} = \boldsymbol{E}\sqrt{\Sigma_x} = \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{(1+\rho)} & -\sqrt{(1-\rho)} \\ \sqrt{(1+\rho)} & \sqrt{(1-\rho)} \end{bmatrix}$$

Table 3 Aircraft in turbulence computations

Frequency response $= Y_j(i\omega)$

Von Karman gust spectrum $= \Phi_g(\omega)$

Current output for durability and damage tolerance

$$\bar{A}_j^2 = \int_{-\infty}^{\infty} |Y_j(i\omega)|^2\Phi_g(\omega)\,\mathrm{d}f$$

$$\bar{V}_j^2 = \int_{-\infty}^{\infty} |Y_j(i\omega)|^2\Phi_g(\omega)\omega^2\,\mathrm{d}f$$

$$N_{0j} = \bar{V}_j/2\pi\bar{A}_j$$

Additional output for PLSGS

$$\sigma_{jk} = \int_{-\infty}^{\infty} Y_j(i\omega)Y_k(-i\omega)\Phi_g(\omega)\,\mathrm{d}f$$

Covariance matrix $= \Sigma_y = [\sigma_{jk}]$

random variables. The generation of pseudo-random normal variates can be done by IMSL subroutines (see Ref. 5).

This gives a procedure to generate from the PSD gust or a taxi covariance matrix a transformation which will impose the required phasing. It can be shown that loads generated by this process satisfy the phasing definition of Eq. (1). An example of this analysis for a phased loading of two loads is given in Table 2. The columns of the T matrix are related to the "phased loads" used in loads analysis for the design and certification of Douglas airplanes. Phased loads and the T matrix have been advocated for use in aircraft design in Ref. 2. The gust analysis computations required as a basis for PLSG are shown in Table 3.

The taxi covariance matrix for the C-17 durability PLSG was estimated from simulations of the nonlinear dynamic aircraft model traveling on prescribed runway forcing functions. Skewness and kurtosis estimates were also computed to detect violation of the Gaussian process assumption. The approximation was deemed adequate for C-17 mission requirements.

Probabilistic Basis For Loading Sequence Generation

The above first step in the PLSG process gives loadings which satisfy the phasing requirement but do not, individually, satisfy the gust exceedance distribution. This defect can be remedied by multiplying each load condition by a random scaling factor. Obtaining the appropriate probability distribution for this scaling factor requires, from durability and damage tolerance theory, the definition of the PSD gust exceedance function

$$N(x) = N_0(P_1 e^{-x/b_1} + P_2 e^{-x/b_2}) \tag{5}$$

where P_1, P_2 are the probabilities of non-storm and storm turbulence, and b_1, b_2 are the nonstorm and storm intensity parameters. The value N_0 is the number of zero crossings with positive slope for the turbulence model. The function $N(x)$ is called in probability theory a "mixed" function. Because of the properties of mixed processes, if analysis is performed for $b_1 = b_2 = b$, $P_1 = 1$, $P_2 = 0$, the results can be easily extended to the general case.

The exceedance curves above are valid only for positive values of load increments x. Negative values are obtained for standard PSD load sequences by assigning a minus sign to odd-numbered values in a randomly generated sequence. This implies a symmetric exceedance distribution. However, it should be noted that any exceedance curve could be used in the following analysis. Thus the phased load sequence generating system can be modified for other than gust environments.

One-Dimensional Case

The preceding two sections provide the foundation for the following analysis. The Monte Carlo generation of load conditions with appropriate phasing properties is specified directly by Eq. (4). A Monte Carlo procedure for generating the scaling variable, however, requires more analysis.

A method for Monte Carlo generation of the peaks and valleys of a standard PSD gust load sequence can be obtained by converting the monotone decreasing gust exceedance distribution into a probability distribution to provide a basis for Monte Carlo generation of a scaling variable. Set

$$\begin{aligned} F(x) &= N(-x)/2N_0, \qquad x < 0, \\ F(x) &= 1 - N(x)/2N_0, \qquad x \geq 0 \end{aligned} \tag{6}$$

When $N(x) = e^{-x/b}$, this gives distribution, density, and characteristic functions. (These functions are concisely described in Ref. 3.)

$$\begin{aligned} F(x) &= \frac{1}{2} e^{x/b} x < 0, \; = 1 - \frac{1}{2} e^{-x/b} x \geq 0 \\ f(x) &= \frac{1}{2b} e^{-|x|/b} \\ \phi(t) &= \int_{-\infty}^{\infty} f(x) e^{ixt}\, dx = 1/(1 + b^2 t^2) \end{aligned} \tag{7}$$

It should be clear that generating both gust peaks and valleys with the same Monte Carlo generator is a trivial departure from the standard method. The standard sequence can be obtained by ordering instead of alternatively assigning plus and minus signs. The above implies that the marginal densities of individual loads in a load condition should be

$$f(x) = \frac{1}{2b} e^{-|x|/b} \tag{8}$$

Let $b = 1$ in the following analysis. This will cause no loss of generality.

Since the random scaling must work for any number N of random loads Y_i in a load condition Y it must also work for $N = 1$. The conditional characteristic function of $V = UY$ given U is

$$\begin{aligned} \phi_{V|U}(t) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-y^2/2u^2 + ity)\, dy \\ &= \exp(-t^2u^2/2) \end{aligned} \tag{9}$$

To obtain the correct distribution for V requires a density function $f_U(u)$ for U such that

$$\begin{aligned} \phi_V(t) &= E\{\phi_{V|U}(t)\} \\ &= \int_0^{\infty} \exp(-t^2u^2/2) f_U(u)\, du \\ &= 1/(1 + t^2) \end{aligned} \tag{10}$$

This integral equation has solution

$$f_U(u) = u e^{-u^2/2} \tag{11}$$

to give

$$\phi_V(t) = \int_0^{\infty} u \exp(-t^2u^2/2) \exp(-u^2/2)\, du = 1/(1 + t^2) \tag{12}$$

Thus a scaling distribution exists so that Monte Carlo generation of a scaling variable can be accomplished. The distribution must apply for all values of N. The cumulative probability distribution for U is

$$F_U(u) = \int_0^u u e^{-u^2/2}\, du = 1 - e^{-u^2/2} \tag{13}$$

Monte Carlo generation of random values of U is accomplished by setting $F(U) = R$ where R is a random variable with uniform distribution on the interval (0, 1) and then solving for U. In this case the equation to be solved is

$$U = \sqrt{-2 \ln (1 - R)} \tag{14}$$

General Case

For the general case let

$$Y^* = \{Y_1, Y_2, \ldots, Y_N\} \tag{15}$$

be a vector of jointly normally distributed random variables with zero mean values and covariance matrix

$$\Sigma = [E\{Y_i, Y_j\}] = [\sigma_{ij}] \tag{16}$$

Define the vector of Fourier transform domain variables

$$t^* \equiv \{t_1, t_2, \ldots, t_N\} \tag{17}$$

and the vector random variables

$$V \equiv UY \tag{18}$$

then the conditional joint characteristic function of V is

$$\phi_{V|U}(t) = \exp(-u^2/2t^*\Sigma t) \tag{19}$$

Applying the scaling density function found in the one dimensional case gives an unconditional characteristic function for V

$$\begin{aligned}\phi_V(t) &= \int_0^\infty \exp(-u^2t^*\Sigma t)u/2 \exp(-u^2/2)\,\mathrm{d}u \\ &= (1 + t^*\Sigma t)^{-1}\end{aligned} \tag{20}$$

Setting $t_i = 0$, $i \neq k$, gives

$$\phi_{Y_k}(t_k) = (1 + \sigma_k^2 t_k^2)^{-1} \tag{21}$$

which is the characteristic function of an exponential distribution with parameter σ_k^2. Hence this generation method yields the correct marginal density functions.

The correlation of Y_j and Y_k can be obtained by differentiation of $\phi_V(t)$. Note that

$$\begin{aligned}-\frac{\partial^2}{\partial t_j \partial t_k}\phi_V(t) &= -\frac{\partial^2}{\partial t_j \partial t_k}E\{\exp(it^*Y)\} \\ &= E\{Y_jY_k\exp(it^*Y)\} \\ &= E\{Y_jY_k\} \quad \text{for} \quad t = 0.\end{aligned} \tag{22}$$

For $\phi_V(t) = (1 + t^*\Sigma t)^{-1}$

$$-\frac{\partial^2}{\partial t_j \partial t_k}\phi_V(t)|_{t=0} = 2\sigma_{jk} \tag{23}$$

so that

$$\begin{aligned}&\text{correlation}(Y_j, Y_k) = 2\sigma_{jk}/\sqrt{(2\sigma_j^2)(2\sigma_k^2)} = \rho_{jk}, \\ &\delta_{jk} = \sqrt{2\sigma_j^2/2\sigma_k^2} = \sigma_j/\sigma_k\end{aligned} \tag{24}$$

which proves that Y has the phasing required by Eq. (1).

The joint density function of V can be expressed in terms of higher order mathematical functions. It is derived here to exhibit its topological properties. It is most readily obtained from the joint conditional density function. The joint conditional characteristic function $\phi_{V/U}(t)$ has Fourier transform

$$f_{V|U}(v) = (2\pi)^{-N/2}|u^2\Sigma|^{-1/2}\exp(-v^*\Sigma^{-1}v/2u^2) \tag{25}$$

which is the joint conditional density function. From this function obtain

$$f_V(v) = \int_0^\infty (2\pi)^{-N/2}[|u^2\Sigma|^{-1/2}\exp(-v^*\Sigma^{-1}v/2u^2)ue^{-u^2/2}]\,\mathrm{d}u \tag{26}$$

where

$$v^* = \{v_1, v_2, \ldots, v_N\} \tag{27}$$

The change of variables $s = u^2/2$ converts this to a known Laplace transform

$$\begin{aligned}f_V(v) &= \int_0^\infty (2\pi)^{-N/2}e^{-s}[|2s\Sigma|^{-1/2}\exp(-s^{-1}v^*\Sigma^{-1}v/4)]\,\mathrm{d}s \\ &= 2(4\pi)^{-N/2}|\Sigma|^{-1/2}[\sqrt{(v^*\Sigma^{-1}v)^{1-N/2}}\,K_{N/2-1}(\sqrt{v^*\Sigma^{-1}v})]\end{aligned} \tag{28}$$

where $K_{N/2-1}$ is the modified Bessel function (see Ref. 6). Note that this density function has the same symmetries as the normal joint density function. It is in fact the n-dimensional analog of the joint normal distribution for the exponential marginal distribution.

Truncating PLSG Sequences

After a standard load sequence is generated, range truncation is usually applied to eliminate values which contribute little to damage or crack growth. Range is defined as the difference between adjacent peak and valley values, hence inherently involves two sequence values. This type of truncation is impossible for phased loading sequences. Basing the truncation on a subset of loads will bias and distort the phasing and the exceedances of the retained load conditions. The only nonbiasing method of truncation is the elliptical truncation, defined below, which eliminates individual load conditions. This method is most nearly akin to the single channel peak truncation. The ASTM definitions of the above truncations are given in Appendix B.

The elliptic (actually hyper-elliptic) truncation requirement is: Eliminate V if

$$V^*\Sigma_y^{-1}V \equiv S < s \tag{29}$$

for some value s chosen so that a specified proportion, p, of load conditions are retained in the truncation process. Proceeding from Eqs. (2), (3), (4), and (18)

$$\begin{aligned}V &= UTX = UE\sqrt{\Sigma_x}X \\ \Sigma_y &= E\Sigma_xE^* \\ E^*E &= EE^* = I \\ \Sigma_y^{-1}Y &= UE\Sigma_x^{-1}E^*E\sqrt{\Sigma_x}X\end{aligned} \tag{30}$$

and

$$\begin{aligned}S &= U^2X^*\sqrt{\Sigma_x}E^*E\Sigma_x^{-1}E^*E\sqrt{\Sigma_x}X \\ &= U^2X^*X\end{aligned} \tag{31}$$

Recall that E is the eigenvector matrix of Σ_y. Thus the inequality $S < s$ can be applied to the normal independent random variables produced for the load sequence generation process to achieve global truncation without involving the transformation T. Thus elliptical truncation does not affect the correlation of loads since it can be applied in the domain of the independent random variables X. The parameter s will be termed the elliptical content in the following discussion.

The effect of elliptical truncation can be dramatically presented, as in Fig. 2, for a load condition consisting of only two loads. A well-defined elliptical region clear of interacting loads is apparent. The boundary of this region is the so-called "equi-probability" boundary for a two-dimensional distribution discussed in Ref. 2. It marks a contour of constant altitude on the probability density function for two loads. Note that the truncation method does not eliminate a small bending moment component if it occurs in conjunction with a large torque component or vice versa.

Application of the truncation requires the determination of the exceedance function for S. This is most easily achieved by first finding the characteristic function of S.

Since the X_i are independently normally distributed and U has the distribution

$$f_U(u) = ue^{-u^2/2} \tag{32}$$

the characteristic function of S for N loads is

$$\begin{aligned}\phi_s(t) &= \int_0^\infty ue^{-u^2/2}(1 - 2itu^2)^{-N/2}\,\mathrm{d}u \\ &= \int_0^\infty e^{-x}(1 - 4itx)^{-N/2}\,\mathrm{d}x\end{aligned} \tag{33}$$

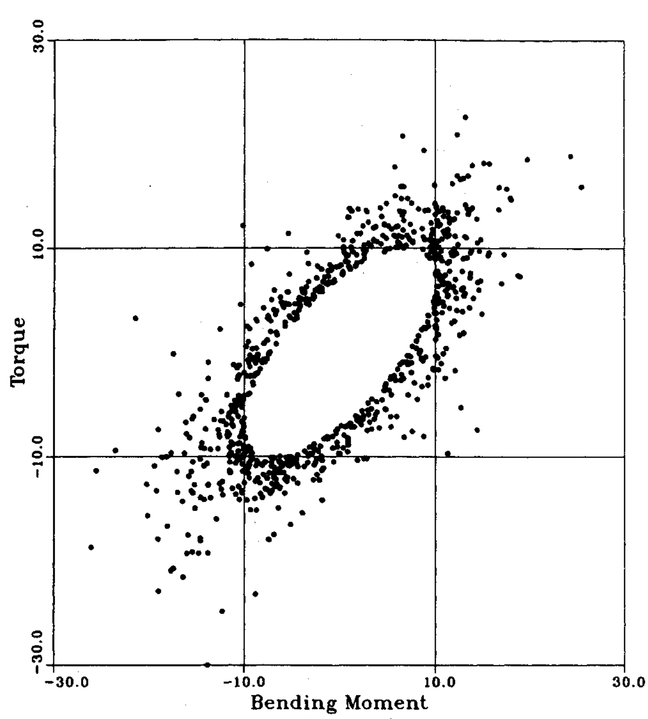

Fig. 2 Scatter after elliptical truncation for two loads.

The moments of S can be found by differentiation of $\phi_s(t)$ for the general case which gives

$$\text{Mean}(S) = 2Nb^2, \qquad \text{Variance}(S) = 4N(N+4)b^4$$
$$E(S^k) = k![(N/2)(N/2+1) \ldots (N/2+k-1)](2b)^{2k} \tag{34}$$

The probability density for S can also be found from $\phi_s(t)$ since

$$\begin{aligned} f(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_s(t)e^{-ist}\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_0^{\infty} e^{-x}(1-4itx)^{-N/2}\,\mathrm{d}x\right]e^{-ist}\,\mathrm{d}t \\ &= \int_0^{\infty} e^{-x}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}(1-4itx)^{-N/2}e^{-ist}\,\mathrm{d}t\right]\mathrm{d}x \\ &= \int_0^{\infty} e^{-x}\left[\frac{\frac{1}{2x}(s/2x)^{N/2-1}e^{-s/4x}}{2^{N/2}\Gamma(N/2)}\right]\mathrm{d}x \\ &= \int_0^{\infty} e^{-s/2\tau}\left[\frac{\frac{1}{2\tau}\tau^{N/2-1}e^{-\tau/2}}{2^{N/2}\Gamma(N/2)}\right]\mathrm{d}\tau \\ &= \frac{1}{N\Gamma(N/2-1)}[(s/4)^{1/2}]^{N/2-1}K_{N/2-1}(\sqrt{s}) \end{aligned} \tag{35}$$

The cumulative distribution function of S is

$$\begin{aligned} F(s) &= \int_0^s f(u)\,\mathrm{d}u \\ &= \int_0^s\int_0^{\infty} e^{-s/2\tau}\left[\frac{\frac{1}{2\tau}\tau^{N/2-1}e^{-\tau/2}}{2^{N/2}\Gamma(N/2)}\right]\mathrm{d}\tau\,\mathrm{d}u \\ &= 1-\int_0^{\infty} e^{-s/2\tau}\left[\frac{\tau^{N/2-1}e^{-\tau/2}}{2^{N/2}\Gamma(N/2)}\right]\mathrm{d}\tau \end{aligned} \tag{36}$$

And, finally, the exceedance function for S is

$$\begin{aligned} e(s) = 1 - F(s) &= \int_0^{\infty} e^{-s/2\tau}\left[\frac{\tau^{N/2-1}e^{-\tau/2}}{2^{N/2}\Gamma(N/2)}\right]\mathrm{d}\tau \\ &= \frac{2}{\Gamma(N/2)}(s/4)^{N/4}K_{N/2}(\sqrt{s}) \end{aligned} \tag{37}$$

In the above K is the modified Bessel function. For the general case ($b \neq 1$) this becomes

$$e(s) = \frac{2}{\Gamma(N/2)}(s/4b^2)^{N/4}K_{N/2}(\sqrt{s}/b) \tag{38}$$

Solving the equation $e(s) = p$, where p is the desired retention proportion, gives the value of s to use in rejecting Monte Carlo generated values UX for which $U^2X^*X < s$.

Truncation Based on a Gust Mission Analysis

The preceding elliptical content exceedance distribution can be most easily used to select content values which give uniform truncation for each flight condition (FC). This, however, gives every flight condition equal importance, even if the flight condition creates no structural damage in the aircraft lifetime.

The problem of accounting for the actual use of an aircraft in the course of its lifetime is usually solved by applying the mission analysis method presented below. Here the amount of time spent in a flight condition becomes an additional parameter in finding design and durability loads. It seems reasonable to apply this approach to PLSGS truncation also. The truncation criteria is then based on the formulation

$$p = \sum_{i=1}^{n} 2\tau_i\left[\frac{P_{1i}}{\Gamma(N/2)}\left(\frac{s_p}{4b_{1i}^2}\right)^{N/4}K_{N/2}\left(\frac{\sqrt{s_p}}{b_{1i}}\right) + \frac{P_{2i}}{\Gamma(N/2)}\left(\frac{s_p}{4b_{2i}^2}\right)^{N/4}K_{N/2}\left(\frac{\sqrt{s_p}}{b_{2i}}\right)\right] \tag{39}$$

where P_{1i} = probability of nonstorm turbulence; P_{2i} = probability of storm turbulence; b_{1i} = intensity of nonstorm turbulence; b_{2i} = intensity of storm turbulence; τ_i = proportion of time in FC segment; p = proportion of retained load conditions; and s_p = constant mission elliptic content for p.

The above formulation is consistent with that used for computing design loads which is

$$2 \times 10^{-5} = \sum_{i=1}^{n} N_{0i}\tau_i[P_{1i}e^{-y/\bar{A}_i b_{1i}} + P_{2i}e^{-y/\bar{A}_i b_{2i}}] \tag{40}$$

where $\bar{A}_i$ = rms load for 1 fps gust; and N_{0i} = number of zero crossings with positive slope.

This design loads formulation, however, accounts for the airplane interaction with the atmosphere through the $\bar{A}$ factor. This can be accomplished for the truncation procedure by computing an average $\bar{A}_i$ for each flight condition. Variation in the $\bar{A}_i$ values for each flight condition is a function of gross weight, configuration, altitude, and velocity effects, etc. The average $\bar{A}$ values are computed as follows.

Let

$$A \equiv [\bar{A}_{ij}] \tag{41}$$

where i, $1 \le i \le m$, are subscripts referring to a subset of rms loads ($\bar{A}$'s) selected from all the external loads of the airplane and j, $1 \le j \le n$, are subscripts referring to all the flight conditions encountered in the lifetime of the airplane.

Define

$$\bar{A}_{i\cdot} \equiv \frac{1}{n}\sum_{j=1}^{n}\bar{A}_{ij} \tag{42}$$

Then

$$w_j \equiv \frac{1}{m}\sum_{i=1}^{m} \bar{A}_{ij}/\bar{A}_{j}. \tag{43}$$

is a nondimensional set of weighting factors accounting empirically for all variations in the flight conditions.

The variation of the $\bar{A}_{ij}$ values about their means $\bar{A}_{i}.$ before and after weighting provides a measure of the weighting effectiveness.

$$V_i \equiv \frac{1}{\bar{A}_{i}.}\sqrt{\frac{1}{n}\sum_{j=1}^{n}(\bar{A}_{ij} - \bar{A}_{j}.)^2}$$

$$a_{ij} \equiv \bar{A}_{ij}/w_j \tag{44}$$

$$v_i \equiv \frac{1}{a_{i}.}\sqrt{\frac{1}{n}\sum_{j=1}^{n}(a_{ij} - a_{i}.)^2}$$

If the weighting by w_j is effective there should generally be a marked decrease in the values of v_i when compared to the V_i.

The truncation equation is modified by the transformation

$$z_p \equiv w_i^2 s_i \tag{45}$$

for each flight condition. The variable z_p measures both elliptical content and the airplane atmospheric interaction. The truncation equation is then

$$p = \sum_{i=1}^{n} 2\tau_i \left[\frac{p_{1i}}{\Gamma(N/2)}\left(\frac{z_p}{4(w_i b_{1i})^2}\right)^{N/4} K_{N/2}\left(\frac{\sqrt{z_p}}{w_i b_{1i}}\right) + \frac{p_{2i}}{\Gamma(N/2)}\left(\frac{z_p}{4(w_i b_{2i})^2}\right)^{N/4} K_{N/2}\left(\frac{\sqrt{z_p}}{w_i b_{2i}}\right)\right] \tag{46}$$

where $z_p \equiv$ constant gust damage content for p. Truncation is applied by solving equation 46 for z_p where again p is the desired retention proportion. The truncation value for a flight condition is computed from $s_i = z_p/w_i^2$ for use in rejecting Monte Carlo generated values UX for which $U^2X^*X < s_i$. (This solution is readily obtained with the Newton-Raphson method.)

Increasing Zero Crossing Variation by Induced Autocorrelation

Use of the PLSGS load conditions in a durability test consists of the simultaneous application of a set of static loads to the aircraft structure. If N load conditions are applied the entire structure is subjected to each load condition so that all parts of the structure have the same load history. Thus the N_0 used in the test is identical for all parts of the structure.

The following technique introduces a limited range of zero crossing variation for the external loads in a load condition. This is an implementation of a concept due to J. Lincoln (see Acknowledgments). The idea is to generate pairs of load conditions in which some loads change while others are the same in both load conditions. The implementation presented here reduces zero crossing for the most strongly correlated loads. If these loads also have low values of N_0, the applied external load sequences will more closely approximate those used in durability and damage tolerance analysis. Autocorrelation is induced between a load condition and its companion by substituting into the successor (immediately following) load condition the scaling factor and the normal variates for the two eigenvectors with the largest eigenvalues (here assumed to have indices 1 and 2). The successor is then replaced by the companion.

$$V_1 = U_1 T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ '' \\ x_n \end{bmatrix} \quad \text{for} \quad V_2 = U_2 T \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ '' \\ \xi_n \end{bmatrix} \tag{47}$$

$$\text{substitute } V_{1c} = U_1 T \begin{bmatrix} x_1 \\ x_2 \\ \xi_3 \\ '' \\ \xi_n \end{bmatrix}$$

The autocovariance between V_1 and V_{1c} is

$$E\{V_1 V_{1c}^*\} = T \begin{bmatrix} 1 & 0 & 0 & '' & 0 \\ 0 & 1 & 0 & '' & 0 \\ 0 & 0 & 0 & '' & 0 \\ '' & '' & '' & '' & '' \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} T^* E\{U_1^2\}$$

The autocorrelation matrix A deriving from the auto covariance has elements

$$a_{ij} = \frac{t_{i1}t_{j1} + t_{i2}t_{j2}}{\sqrt{\left[\sum_{k=1}^{n} t_{jk}^2\right]\left[\sum_{k=1}^{n} t_{jk}^2\right]}} \tag{48}$$

If a_{ii} is close to 1 external load i will tend to have the same value in the companion load condition as it had in the initial load condition.

The phasing for V_{1c} is preserved since

$$E\{V_{1c} V_{1c}^*\} = E\Sigma_x E^* E\{U_1^2\} = \Sigma_y E\{U_1^2\} \tag{49}$$

The exceedance function for V_{1c} is also preserved. However, the use of the scaling variable of the initial load in the companion load must be accounted for if truncation is applied.

Conclusions

A general system for generating phased loading sequences of gust or taxi loads at multiple loading stations has been presented. This technique will provide a set of simultaneously acting random loads such that the marginal probability distribution for each load is consistent with gust or taxi exceedance functions and possessing proper correlations for all combinations of loads. The process uses a covariance matrix of loads, computed in PSD gust or taxi analysis, to generate Gaussian phased loadings. Each loading is multiplied by a random scaling variable to make any load in the phased loading sequence satisfy the gust or taxi exceedance curve for that load. If the dynamic model of the aircraft produces fundamentally balanced loads, then any member of the multiple loads durability sequence will also be balanced. This greatly eases testing of large durability articles, where any imbalance of loading can produce disastrous repercussions.

Truncation of loadings is done by an "elliptical" method which, unlike standard range truncation methods, will not bias and distort the phasing and exceedances of the retained load conditions. This method of truncation also accounts for the relative severity of flights in the mission profile of the aircraft.

The method inherently results in the same number of zero crossings for all loads in the load condition. This can be modified by an "induced autocorrelation" approach which causes selected loads in a load condition to hold approximately constant for two successive applications.

The method will be implemented for full scale and component C-17 aircraft durability testing during 1992.

Appendix A: Recurrence Relationship for the Elliptical Content Exceedance Function

Additional analysis is required to order to compute the exceedance function for elliptical content for large values of N, the number of independent normal variables used in the phased load sequence generation. The modified Bessel function satisfies the recurrence relationship

$$K_{n+1}(x) = \frac{2n}{n} K_n(x) + K_{n-1}(x) \tag{A1}$$

Define, using Eq. (38),

$$e(s) \equiv \psi_{N/2}(\sqrt{s}/b)$$
$$\psi_m(x) \equiv \frac{2}{\Gamma(m)} (x/2)^m K_m(x) \tag{A2}$$

Then

$$\psi_{m+1}(x) \equiv \frac{2}{\Gamma(m+1)} (x/2)^{m+1} \left[\frac{2m}{x} K_m(x) + K_{m-1}(x)\right] \tag{A3}$$
$$= \psi_m(x) + \frac{1}{m(m-1)} (x/2)^2 \psi_{m-1}(x)$$

All of the truncation exceedance functions presented here must be solved implicity. Hence it is noted, for use in Newton-Raphson applications,

$$e'(s) = -\frac{1}{2N} \psi_{N/2-1}(\sqrt{s}/b) \tag{A4}$$

Appendix B: Glossary of Terms

Covariance The covariance of two random variables X_i, X_j is defined in terms of the moments of the joint probability density function of X_i and X_j:

$$\mu_i = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i f(x_i, x_j)\, \mathrm{d}x_i\, \mathrm{d}x_j$$
$$\sigma_i^2 \equiv \sigma_{ii} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_i - \mu_i)^2 f(x_i, x_j)\, \mathrm{d}x_i\, \mathrm{d}x_j \tag{B1}$$
$$\sigma_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_i - \mu_i)\,(x_j - \mu_j) f(x_i, x_j)\, \mathrm{d}x_i\, \mathrm{d}x_j$$

Covariance matrix The matrix Σ with elements σ_{ij}, $\Sigma \equiv [\sigma_{ij}]$ is called the covariance matrix of the set of random variables $\{X_i, 1 \leq i \leq n\}$

Standard load sequence A spectrum loading in which peaks and valleys alternate

Exceedance spectrum Representation of spectrum loading contents by the number of times specified values of a particular loading parameter (peak, range, etc.) are equaled or exceeded (see Ref. 7)

Gust exceedance distribution This is a gust **exceedance spectrum**

Load condition or **loading** Multiple loads simultaneously applied at several points on a structure

Load sequence See **spectrum loading**

Marginal distribution The distribution for a single random variable of a collection of jointly distributed variables (see Ref. 4)

Monte Carlo method The Monte Carlo method for generating random numbers is based on the fact that the cumulative distribution function (cdf), $F(x)$, of a random variable x is a monotone increasing function, $0 \leq F(x) \leq 1$. If $F(x)$ is also continuous the transformation

$$x = F^{-1}(y), \qquad 0 \leq y \leq 1 \tag{B2}$$

is a one to one mapping of the interval [0, 1] onto to a subinterval of $(-\infty, \infty)$. Even if $F(x)$ is not continuous the transformation can be defined to carry intervals of [0, 1] onto to values in $(-\infty, \infty)$. Thus the cdf provides a means of transforming random variables Y with a uniform distribution into random variables X with cdf $F(x)$. In practice Y is usually a "pseudorandom" number generated with a computer algorithm.

Peak The point at which the first derivative of the load-time history changes from a positive to a negative sign (see Ref. 7)

Range The algebraic difference between successive valley and peak loads (positive range or increasing load range) or between successive peak and valley loads (negative range or decreasing load range) (see Ref. 7)

Single channel sequence A single **spectrum loading** applied in test or analysis

Spectrum loading A loading in which all the peak loads are not equal or all the valley loads are not equal, or both (see Ref. 7)

Truncation The exclusion of cycles with values above, or the exclusion of cycles with values below a specified level (referred to as truncation level) of a loading parameter (peak, valley, range, etc.) (see Ref. 7)

Valley The point at which the first derivative of the load-time history changes from a negative to a positive sign (see Ref. 7)

Zero crossing N_0 or n_0 the number of times that a load-time history crosses zero load level with a positive slope (or a negative slope, or both, as specified) during a given length of history. In this publication the positive slope definition is used (see Ref. 7)

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